Problem 3.2

(a) For what range of ν is the function $f(x) = x^\nu$ in Hilbert space, on the interval $(0, 1)$? Assume ν is real, but not necessarily positive.

(b) For the specific case $\nu = 1/2$, is $f(x)$ in this Hilbert space? What about $xf(x)$? How about $(d/dx)f(x)$?

Solution

A function $f(x)$ is said to be in Hilbert space if it's square-integrable on a given interval.

$$\int_a^b |f(x)|^2\,dx < \infty$$

Part (a)

For $f(x) = x^\nu$ on the interval $0 < x < 1$,

$$\begin{aligned}
\int_a^b |f(x)|^2\,dx &= \int_0^1 (x^\nu)^2\,dx \\
&= \int_0^1 x^{2\nu}\,dx \\
&= \begin{cases} \left.\dfrac{x^{2\nu+1}}{2\nu+1}\right|_0^1 & \text{if } 2\nu \neq -1 \\[2ex] \left.\ln x\right|_0^1 & \text{if } 2\nu = -1 \end{cases} \\
&= \begin{cases} \dfrac{1}{2\nu+1}(1^{2\nu+1} - 0^{2\nu+1}) & \text{if } 2\nu \neq -1 \\[2ex] \ln 1 - \ln 0 & \text{if } 2\nu = -1 \end{cases} \\
&= \begin{cases} \dfrac{1}{2\nu+1} & \text{if } 2\nu + 1 > 0 \\[2ex] \infty & \text{if } 2\nu + 1 < 0 \\[2ex] 0 - (-\infty) & \text{if } 2\nu = -1 \end{cases} \\
&= \begin{cases} \dfrac{1}{2\nu+1} & \text{if } 2\nu > -1 \\[2ex] \infty & \text{if } 2\nu \leq -1 \end{cases}.
\end{aligned}$$

Therefore, on the interval $0 < x < 1$, $f(x) = x^\nu$ is in Hilbert space if $\nu > -1/2$.

Part (b)

For the specific case that $\nu = 1/2$, $f(x) = x^{1/2}$.

$$\int_0^1 |f(x)|^2\,dx = \int_0^1 (x^{1/2})^2\,dx = \int_0^1 x\,dx = \frac{1}{2} < \infty$$

$$\int_0^1 |xf(x)|^2\,dx = \int_0^1 (x^{3/2})^2\,dx = \int_0^1 x^3\,dx = \frac{1}{4} < \infty$$

$$\int_0^1 \left|\frac{d}{dx}f(x)\right|^2 dx = \int_0^1 \left(\frac{1}{2}x^{-1/2}\right)^2 dx = \frac{1}{4}\int_0^1 x^{-1}\,dx = \infty$$

Therefore, on the interval $0 < x < 1$, $f(x)$ and $xf(x)$ are in Hilbert space, but $(d/dx)f(x)$ is not in Hilbert space.